



**W<sup>2</sup> Education**

This document reflects the typical format of an official HSC paper, complete with solutions and supplementary comments. While originally created to support our own students, it is now made available to a wider audience, offering valuable insights for anyone seeking to improve their approach.

**2026** COMPREHENSIVE EXAM REVIEW – DETAILED SOLUTIONS & COMMENTS

# Mathematics Extension 1

## General Comments

- Original examination items are shown in black
- Answers and suggested solutions are shown in red
- Comments and postmortem insights are shown in blue
- The formatting of questions may NOT be identical to the primary source
- This document features suggested solutions which do NOT serve as official solutions for this paper
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- For any enquiries or concerns, please refer to the admin contact details at the end of this document

## Total marks: 70

### Section I – 10 marks (pages 2–10)

- Questions 1–10 with Answers
- Allow about 15 minutes for this section

### Section II – 60 marks (pages 11–31)

- Questions 11–14 with Solutions
- Allow about 1 hour and 45 minutes for this section

## Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the maximum value of  $\sin \alpha \cos \alpha$ ?

A.  $\frac{1}{2}$

B. 1

C.  $\sqrt{2}$

D. 2

As  $\sin \alpha \cos \alpha = \frac{1}{2} \sin 2\alpha$  and  $\sin 2\alpha \leq 1$ , the maximum value is  $\frac{1}{2}$ .

2 Given  $y = \sin^{-1}\left(\frac{2x}{3}\right)$ , what is  $\frac{dy}{dx}$ ?

A.  $\frac{2}{3} \cos^{-1}\left(\frac{2x}{3}\right)$

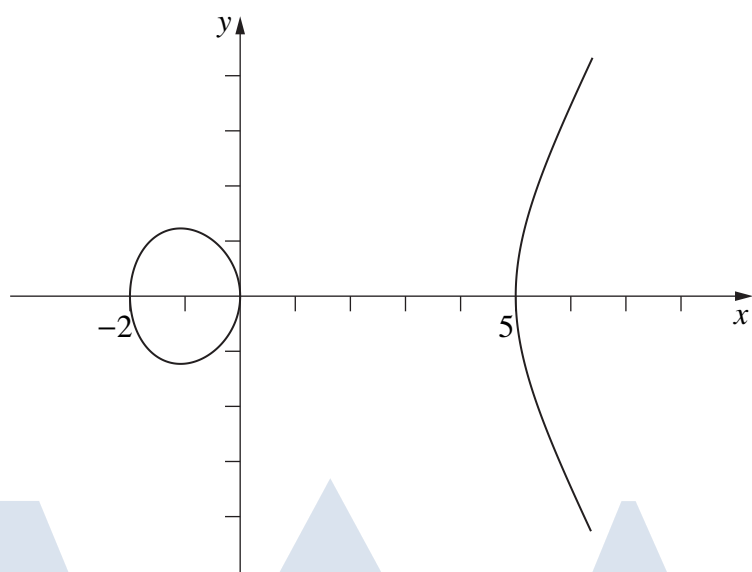
B.  $-\frac{2}{3} \cos^{-1}\left(\frac{2x}{3}\right)$

C.  $\frac{2}{3\sqrt{1-4x^2}}$

D.  $\frac{2}{\sqrt{9-4x^2}}$

The derivative approaches infinity and is undefined at  $x = \frac{3}{2}$ .

- 3 Consider the graph of  $y^2 = f(x)$  as shown.



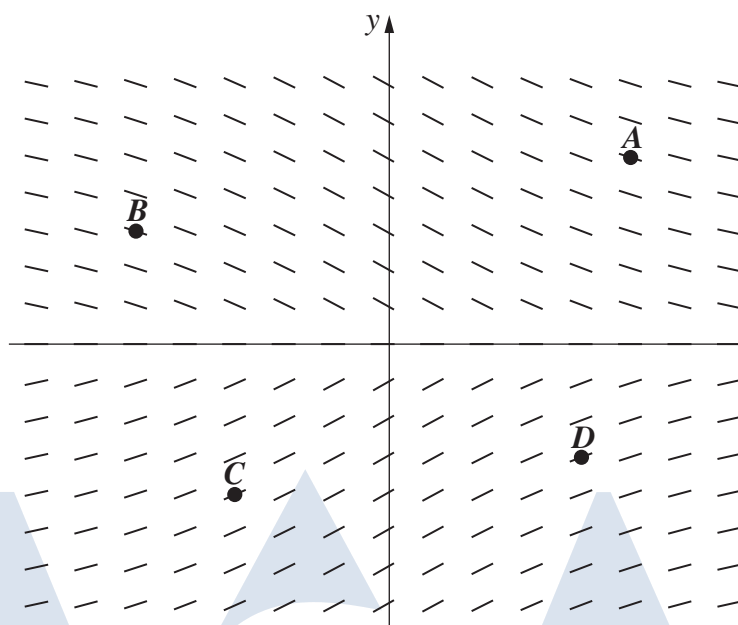
What is a possible equation for  $f(x)$ ?

- A.  $f(x) = -x(x + 2)(x - 5)$
- B.  $f(x) = -x(x - 2)(x + 5)$
- C.  $f(x) = x(x + 2)(x - 5)$
- D.  $f(x) = x(x - 2)(x + 5)$

We need  $f(x) > 0$  for  $x \in (-2, 0) \cup (5, \infty)$  and  $f(x) = 0$  at  $x = -2, 0, 5$ .

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- 4 A direction (slope) field representing a differential equation is shown.



A function is increasing and concave down at a point. If the function is a solution to the differential equation, which of the following could be the point?

- A. Point A
- B. Point B
- C. Point C
- D. Point D

We need  $\frac{dy}{dx} > 0$ , which occurs at points *C* and *D*. At *C*, the curve has positive concavity.

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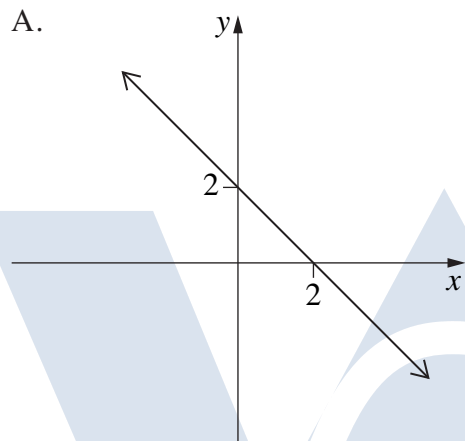
- 5 A function is defined by the set of parametric equations

$$\begin{aligned}x &= t^2 - 3 \\ y &= 5 - t^2\end{aligned}$$

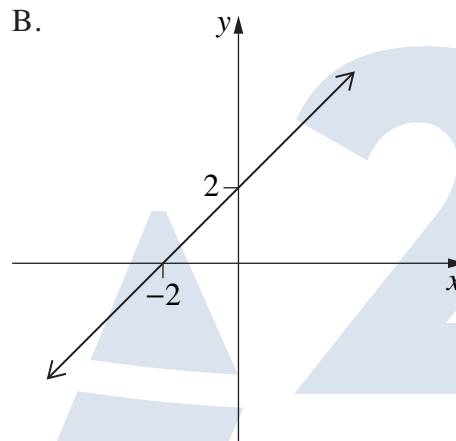
where  $t$  is a real number.

Which of the following graphs best represents the function?

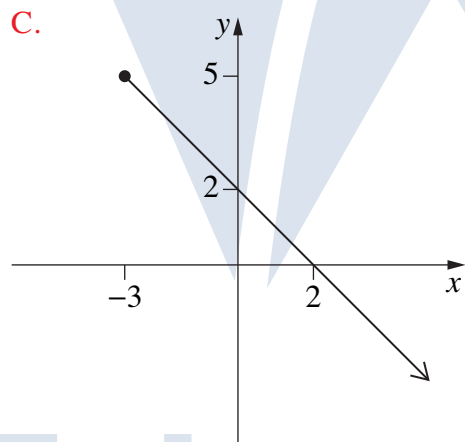
A.



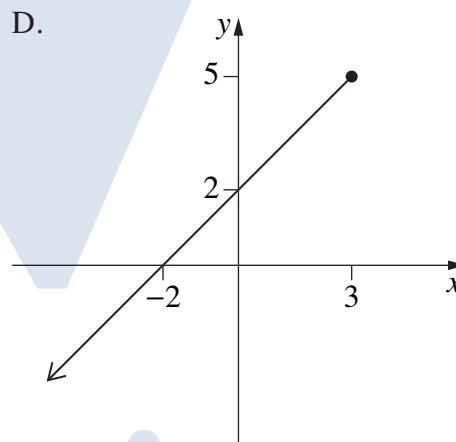
B.



C.



D.



From  $x = t^2 - 3$ , we need  $x \geq -3$ .

- 6 Let  $f(x) = x^3 - 7x$  and  $g(x) = f^{-1}(x)$ , where  $g(-6) = 2$ .

What is the value of  $g'(-6)$ ?

- A.  $\frac{1}{101}$
- B.  $\frac{1}{5}$**
- C. 5
- D. 101

As  $g(-6) = 2$ ,  $f(2) = -6$  so  $g'(-6) = \frac{1}{f'(2)} = \frac{1}{3(2^2) - 7} = \frac{1}{5}$ .

Unrestricted,  $f$  is NOT one-to-one, and thus  $f$  does NOT admit an inverse function on its whole domain. The question implies a restricted domain such as  $x \geq \sqrt{\frac{7}{3}}$ , which contains  $x = 2$ .

A question on the derivative of an inverse function at a point was predicted in the multiple-choice section of our **2026 HSC Masterclass** HSC-level practice paper.

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7 In a false proof by induction for integers  $n \geq 1$ ,  $S(n)$  is the statement:

In every list containing  $n$  numbers, all the numbers are equal to each other.

So  $S(1)$  is the statement:

In every list containing 1 number, that number is equal to itself.

$S(2)$  is the statement:

In every list containing 2 numbers, those numbers are equal.

Which of the following is FALSE?

- A.  $S(1)$
- B. If  $S(1)$  then  $S(2)$
- C. If  $S(2)$  then  $S(3)$
- D. If  $S(3)$  then  $S(4)$

We can confirm that  $S(1)$  is true and  $S(2)$  is false. That is,  $S(1)$  being true does NOT imply that  $S(2)$  is true.

# Education

- 8 The position vector of a particle at time  $t$  is given by

$$\underline{r}(t) = (b - \sqrt{a} \cos t) \underline{i} + \left(a - \frac{1}{b} \sin t\right) \underline{j}$$

where  $a$  and  $b$  are positive real numbers.

If the path of the particle is a circle, which of the following expressions will be equal to 1?

A.  $\frac{\sqrt{b}}{a}$

B.  $\frac{a}{b^2}$

C.  $a\sqrt{b}$

D.  $ab^2$

For the path to be a circle,  $|\sqrt{a}| = \left|-\frac{1}{b}\right|$ . Squaring gives  $a = \frac{1}{b^2}$ .

Education

- 9 Socks are randomly selected from a basket containing one odd sock and  $n$  matching pairs of socks.

Selecting 7 socks at random guarantees a matching pair of socks. Selecting 6 socks at random does not guarantee a matching pair of socks.

What is the value of  $n$ ?

- A. 5
- B. 6
- C. 11
- D. 12

The basket holds  $n + 1$  different kinds of sock, so at most  $n + 1$  socks can be selected without a matching pair.

As 6 socks do not guarantee a pair,  $n + 1 \geq 6$ . But 7 socks do, so  $n + 1 \leq 6$ .

Hence  $n + 1 = 6$ , and so  $n = 5$ .

Alternatively, we can affirm A. by considering the pigeonhole principle on the scenario  $0, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$  ( $n = 5$ ).

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- 10 In a lottery game, a barrel contains 46 balls that are numbered 1 to 46. Ten balls are randomly drawn from the barrel. The order of the draw does not matter and drawn balls are not replaced. A player chooses six numbers. A prize is won if all of the numbers chosen by the player match numbers on the balls drawn from the barrel.

What is the probability that the player wins a prize?

A.  $\frac{{}^{40}C_4}{{}^{46}C_{10}}$

B.  $\frac{{}^{40}C_6}{{}^{46}C_{10}}$

C.  $\frac{{}^{46}C_4}{{}^{46}C_{10}}$

D.  $\frac{{}^{46}C_6}{{}^{46}C_{10}}$

Out of  ${}^{46}C_{10}$  possible draws, the number that win a prize is  ${}^{40}C_4$ , which is the number of ways the four balls which are NOT within the player's selections can be chosen.

#### Alternative method

With  $n = 46$  balls,  $k = 10$  drawn and  $r = 6$  chosen, A. is  $\frac{{}^{n-r}C_{k-r}}{{}^nC_k}$ . If all balls are drawn so that  $k = n$ , the player certainly wins, and A. becomes  $\frac{{}^{n-r}C_{k-r}}{{}^nC_k} = 1$ , while B., C. and D. do NOT equal 1 in general.

Hence, we eliminate B., C. and D..

#### Alternative method (credit to our student ZL)

Consider the player's six numbers one at a time. The first is among the ten balls drawn with probability  $\frac{10}{46}$ . Given this, the second is among the other nine balls drawn with probability  $\frac{9}{45}$ , and so on.

So the probability is  $\frac{10}{46} \times \frac{9}{45} \times \frac{8}{44} \times \frac{7}{43} \times \frac{6}{42} \times \frac{5}{41} = \frac{10}{446\,039} = \frac{{}^{40}C_4}{{}^{46}C_{10}}$ .

## Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

Answer the questions in the Section II Writing Booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Answer Question 11 on pages 2–5 of the Writing Booklet.

- (a) When the polynomial  $P(x)$  is divided by  $9x^2 - 1$  the remainder is  $3x + 7$ . 2

Find the remainder when  $P(x)$  is divided by  $3x - 1$ .

Write  $P(x) \equiv (3x - 1)(3x + 1)Q_1(x) + 3x + 7$ , where  $Q_1(x)$  is the quotient when  $P(x)$  is divided by  $9x^2 - 1$ .

Divide  $P(x)$  by  $3x - 1$  by writing  $P(x) \equiv (3x - 1)Q_2(x) + R$ , where  $Q_2(x)$  is the quotient and  $R$  is the constant remainder, as  $3x - 1$  is linear.

Put  $x = \frac{1}{3}$ , so  $R = P\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) + 7$ .

Hence the remainder is 8.

### Alternative method

As  $3x + 7 = (3x - 1) + 8$ ,  $P(x) \equiv (3x - 1)((3x + 1)Q_1(x) + 1) + 8$ .

Hence, as 8 is a constant and  $3x - 1$  is linear, the remainder is 8.

Though slightly imprecise, it probably would have been okay to use the remainder theorem with the *non-monic* linear factor  $(3x - 1)$ .

Question 11 (continued)

(b) Solve  $\frac{x+6}{x-3} > 4$ .

3

Consider  $\frac{x+6}{x-3} = \frac{(x-3)+9}{x-3} > 4$

$$1 + \frac{9}{x-3} > 4$$

$$\frac{9}{x-3} > 3 > 0, \text{ so } x-3 > 0, \text{ i.e., } x > 3.$$

As  $x-3 > 0$ , multiplying by  $(x-3)$ ,

$$9 > 3(x-3)$$

$$x < 6.$$

Hence  $x \in (3, 6)$ .

**Alternative method**

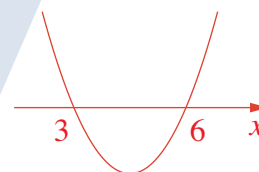
As  $(x-3)^2 > 0$  for  $x \neq 3$ , multiply both sides by  $(x-3)^2$ .

So

$$(x+6)(x-3) > 4(x-3)^2$$

$$(x-3)(4x-12-x-6) < 0$$

$$3(x-3)(x-6) < 0$$



Hence  $x \in (3, 6)$ .

Education

Question 11 (continued)

- (c) For the function  $y = 2\sin^{-1}\left(\frac{x}{3}\right)$ , find the domain and range.

2

For  $y(x)$ , both  $-1 \leq \frac{x}{3} \leq 1$  and  $-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$ .

So  $-3 \leq x \leq 3$  and  $-\pi \leq y \leq \pi$ .

Hence the domain is  $[-3, 3]$  and the range is  $[-\pi, \pi]$ .

- (d) The vectors  $\begin{pmatrix} m \\ -8 \end{pmatrix}$  and  $\begin{pmatrix} m-2 \\ 1 \end{pmatrix}$  are parallel.

Find  $m$ .

For  $\begin{pmatrix} m \\ -8 \end{pmatrix} \parallel \begin{pmatrix} m-2 \\ 1 \end{pmatrix}$ ,

$$\begin{aligned}\frac{m-2}{m} &= 1 - \frac{2}{m} = \frac{1}{-8}, \quad m \neq 0 \\ m &= \frac{2}{1 + \frac{1}{8}} \\ &= \frac{16}{9}.\end{aligned}$$

Hence  $m = \frac{16}{9}$ .

**Alternative method**

As the vectors are parallel,  $\begin{pmatrix} m \\ -8 \end{pmatrix} = \lambda \begin{pmatrix} m-2 \\ 1 \end{pmatrix}$ , for some real  $\lambda$ .

$$\begin{aligned}\text{So } \lambda &= -8 \text{ and } m = -8(m-2) \\ &= -8m + 16 \\ 9m &= 16.\end{aligned}$$

Hence  $m = \frac{16}{9}$ .

Question 11 (continued)

(e) Solve  $2\cos^3 x - 2\cos^2 x - \cos x + 1 = 0$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .

3

Consider  $2\cos^3 x - 2\cos^2 x - \cos x + 1 = 2\cos^2 x(\cos x - 1) - (\cos x - 1)$   
 $= (\cos x - 1)(2\cos^2 x - 1).$

So  $\cos x = 1$  or  $\cos 2x = 0$ .

Thus  $x = 0$  or  $2x = \pm \frac{\pi}{2}$ , for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .

Hence  $x = -\frac{\pi}{4}, 0, \frac{\pi}{4}$ .

W<sup>2</sup> Education

Question 11 (continued)

- (f) Evaluate  $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}} dx$  using the substitution  $u = x^2$ .

3

With  $u = x^2$ ,  $du = 2x dx$ .

At  $x = 0$ ,  $u = 0$ , and at  $x = \frac{1}{\sqrt{2}}$ ,  $u = \frac{1}{2}$ .

So

$$\begin{aligned}\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}} dx &= \frac{1}{2} \int_0^{\frac{1}{\sqrt{2}}} \frac{2x dx}{\sqrt{1-(x^2)^2}} \\ &= \frac{1}{2} \int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} \\ &= \frac{1}{2} \sin^{-1} u \Big|_0^{\frac{1}{2}} \\ &= \frac{1}{2} \left( \frac{\pi}{6} - 0 \right) \\ &= \frac{\pi}{12}.\end{aligned}$$

This exact definite integral was set as Question 11 (b) of our **2026 Trial Premiere**.

Education

**Question 12** (15 marks) Answer Question 12 on pages 7–10 of the Writing Booklet.

- (a) In a particular population, the percentage of people with blood type B+ is 12%. **2**  
A blood bank takes blood from 20 people one morning.

Find the probability that at least two of the people have blood type B+, correct to 3 decimal places.

Let  $X$  be the number of the 20 people with blood type B+, so  $X \sim \text{Bin}(20, 0.12)$ .

Then

$$\begin{aligned}\Pr(X \geq 2) &= 1 - \Pr(X = 0) - \Pr(X = 1) \\ &= 1 - (0.88)^{20} - 20(0.12)(0.88)^{19} \\ &= 0.7109\dots \\ &\approx 0.711 \text{ (3 d.p.)}.\end{aligned}$$

- (b) Prove by mathematical induction that  $n(n^2 + 2)$  is divisible by 3, for all integers  $n \geq 1$ . **3**

For  $n = 1$ ,  $n(n^2 + 2) = 1(1 + 2) = 3$ , which is divisible by 3.

The statement is true for  $n = 1$ .

Assume true for  $n = k$ , where  $k$  is a positive integer, so

$$k(k^2 + 2) = 3P, \text{ where } P \in \mathbb{Z}$$

to prove also true for  $n = k + 1$ , i.e.,

$$(k + 1)(k^2 + 2k + 3) = 3Q, \text{ where } Q \in \mathbb{Z}.$$

$$\begin{aligned}\text{Consider } (k + 1)(k^2 + 2k + 3) &= (k^3 + 2k^2 + 3k + 3) \\ &= 3P + 3(k^2 + k + 1), \text{ using the assumption} \\ &= 3(P + k^2 + k + 1) \\ &= 3Q, \text{ where } Q = P + k^2 + k + 1 \in \mathbb{Z}, \text{ as } P, k \in \mathbb{Z}.\end{aligned}$$

Hence the statement is true by mathematical induction for  $n \geq 1$ .

Question 12 (continued)

- (c) Given  $\vec{a} \cdot \vec{b} = 3$ ,  $|\vec{a}| = 2$  and  $|\vec{b}| = 3$ , find  $|\vec{a} - 2\vec{b}|$ .

2

Consider

$$\begin{aligned} |\vec{a} - 2\vec{b}|^2 &= (\vec{a} - 2\vec{b}) \cdot (\vec{a} - 2\vec{b}) \\ &= \vec{a} \cdot \vec{a} - \vec{a} \cdot (2\vec{b}) - (2\vec{b}) \cdot \vec{a} + 4\vec{b} \cdot \vec{b} \\ &= |\vec{a}|^2 - 4(\vec{a} \cdot \vec{b}) + 4|\vec{b}|^2 \\ &= 4 - 4(3) + 4(9) \\ &= 2^2(7). \end{aligned}$$

Hence  $|\vec{a} - 2\vec{b}| = 2\sqrt{7}$ , as  $|\vec{a} - 2\vec{b}| \geq 0$ .

A near-identical question appeared in our **2024 HSC Premiere** (Question 11 (e)), differing only in that both vectors were unit vectors at  $60^\circ$  and the expansion of  $|\vec{u} - \vec{v}|^2$  was given.

- (d) Solve  $\tan^{-1}(5x) - \tan^{-1}(3x) = \tan^{-1}\left(\frac{1}{4}\right)$ .

2

With  $\alpha = \tan^{-1}(5x)$  and  $\beta = \tan^{-1}(3x)$ , solve  $\alpha - \beta = \tan^{-1}\left(\frac{1}{4}\right)$ , where  $\tan \alpha = 5x$  and  $\tan \beta = 3x$ .

Then

$$\tan(\alpha - \beta) = \frac{5x - 3x}{1 + (5x)(3x)} = \frac{1}{4}$$

$$\frac{2x}{1 + 15x^2} = \frac{1}{4}$$

$$15x^2 - 8x + 1 = 0$$

$$(1 - 3x)(1 - 5x) = 0.$$

Hence  $x = \frac{1}{3}$  or  $x = \frac{1}{5}$ .

A very common question type drilled countless times in exam training class tests. In fact this HSC question is one of the simplest types because no extraneous solutions were introduced by a naïve method.

Question 12 (continued)

- (e) The angle between the vectors  $\begin{pmatrix} m \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$  is  $\frac{\pi}{4}$ .

3

Find the value(s) of  $m$ .

$$\text{As } \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\begin{pmatrix} m \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\sqrt{m^2 + 4}\sqrt{10}},$$

$$10(m^2 + 4) = 2(m + 6)^2, \quad m \geq -6$$

$$4m^2 - 12m - 16 = 0$$

$$4(m - 4)(m + 1) = 0.$$

Hence  $m = 4$  or  $m = -1$ .

A near-identical question was predicted in our **2026 HSC Masterclass** HSC-level practice paper (Question 11 (b)), but a more conceptually layered question since one root of the resulting quadratic had to be rejected.

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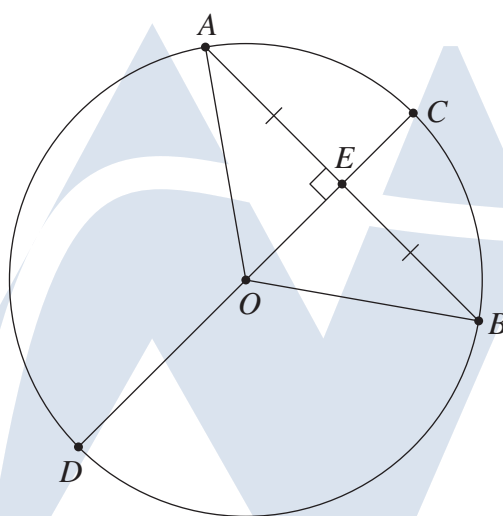
Question 12 (continued)

- (f) The points  $A, B, C$  and  $D$  lie on the circumference of the circle centred at  $O$  with radius 5.

The position vector  $\overrightarrow{OA}$  is  $\begin{pmatrix} 3\cos\theta - 4\sin\theta \\ 3\sin\theta + 4\cos\theta \end{pmatrix}$ .

The position vector  $\overrightarrow{OB}$  is  $\begin{pmatrix} 3\cos\theta + 4\sin\theta \\ 3\sin\theta - 4\cos\theta \end{pmatrix}$ .

The diameter  $\overline{CD}$  is the perpendicular bisector of  $\overline{AB}$ , which intersects it at  $E$ .



- (i) Show that  $\overrightarrow{OE} = 3 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ .

1

As  $E$  is the midpoint of  $AB$ ,

$$\begin{aligned} \overrightarrow{OE} &= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA}) \\ &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) \\ &= \frac{1}{2} \begin{pmatrix} 6\cos\theta \\ 6\sin\theta \end{pmatrix} \\ &= 3 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \end{aligned}$$

Question 12 (continued)

(f) (continued)

(ii) Hence show that  $|\overrightarrow{AE}|^2 = |\overrightarrow{CE}| |\overrightarrow{DE}|$ .

2

As  $|\overrightarrow{OC}| = 5$  (radius) and  $\overrightarrow{OC} \cdot \overrightarrow{OE} > 0$ ,  $\overrightarrow{OC} = 5 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ , so  $\overrightarrow{OD} = -\overrightarrow{OC} = -5 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ .

As  $\overrightarrow{AE} = \begin{pmatrix} 3 \cos \theta \\ 3 \sin \theta \end{pmatrix} - \begin{pmatrix} 3 \cos \theta - 4 \sin \theta \\ 3 \sin \theta + 4 \cos \theta \end{pmatrix} = 4 \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ ,  $|\overrightarrow{AE}| = 4$ , as  $\sqrt{\sin^2 \theta + \cos^2 \theta} = 1$ .

Now

$$\overrightarrow{CE} = 3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} - 5 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = -2 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\overrightarrow{DE} = 3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + 5 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = 8 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}.$$

$$\text{Hence } |\overrightarrow{CE}| |\overrightarrow{DE}| = 2 \times 8 = 16 = 4^2 = |\overrightarrow{AE}|^2.$$

**Alternative method**

By (i),  $|\overrightarrow{OE}| = 3$ . As  $E$  lies on the diameter  $CD$  and  $3 < 5$ , by the diagram  $|\overrightarrow{CE}| = 5 - 3 = 2$  and  $|\overrightarrow{DE}| = 5 + 3 = 8$ . As  $CD \perp AB$ , triangle  $OEA$  is right-angled at  $E$ , so by Pythagoras' theorem,

$$\begin{aligned} |\overrightarrow{AE}|^2 &= |\overrightarrow{OA}|^2 - |\overrightarrow{OE}|^2 \\ &= 5^2 - 3^2 = 16 = 2 \times 8 = |\overrightarrow{CE}| |\overrightarrow{DE}|. \end{aligned}$$

Education

**Question 13** (15 marks) Answer Question 13 on pages 12–15 of the Writing Booklet.

- (a) A multiple-choice exam has  $n$  questions, each with four possible answers.

3

It is assumed that the probability of guessing the correct answer is  $p = \frac{1}{4}$ , and the sample proportion  $\hat{p}$  is approximately normally distributed with  $\mu = p$  and  $\sigma^2 = \frac{p(1-p)}{n}$ .

In order to pass the exam, a student must correctly answer at least 20% of the questions.

If the student guesses every answer, find the smallest value of  $n$  which gives them a 99.01% chance of passing the exam.

You may use the standard normal distribution and the information on page 15.

Let  $X$  be the number of correct guesses, so  $X \sim \text{Bin}\left(n, \frac{1}{4}\right)$ . Then  $\hat{p} = \frac{X}{n}$  follows approximately  $N\left(\frac{1}{4}, \left(\sqrt{\frac{3}{16n}}\right)^2\right)$ .

For  $\Pr(\hat{p} \geq 0.2) \approx \Pr\left(Z \geq \frac{0.2 - 0.25}{\sqrt{3}} \times 4\sqrt{n}\right) \geq 0.9901$ ,  
 $\Pr\left(Z \leq \frac{4(0.05)}{\sqrt{3}}\sqrt{n}\right) \geq 0.9901$ .

By the table,  $0.2\sqrt{\frac{n}{3}} \geq 2.33 > 0$ , so  $n \geq 3\left(\frac{2.33}{0.2}\right)^2 = 407.1675$ .

Hence the smallest such  $n$  is 408.

The wording of the question is ambiguous. The value of  $n$  that yields the closest numerical probability to 99.01% is  $n = 407$ . Yet the word ‘smallest’ suggests that the objective is to find  $n$  which yields *at least* a 99.01% chance of passing.

Question 13 (continued)

- (b) Prove  $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}$ , where  $m$  and  $n$  are positive integers. 2

$$\begin{aligned}\text{Consider } (m+n)(m+n-1) &= m^2 + 2mn + n^2 - m - n \\ &= m(m-1) + n(n-1) + 2mn.\end{aligned}$$

$$\text{So } \frac{(m+n)(m+n-1)}{2} = \frac{m(m-1)}{2} + \frac{n(n-1)}{2} + mn.$$

$$\text{Hence } \binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}, \text{ as } m, n \in \mathbb{Z}^+.$$

**Alternative method**

Consider choosing 2 kids from a group of  $m+n$  kids, consisting of  $m$  boys and  $n$  girls. This can be done in  $\binom{m+n}{2}$  ways.

Consider the possible gender(s):

For 2 boys, there are  $\binom{m}{2}$  ways.

For 1 boy and 1 girl, there are  $\binom{m}{1}\binom{n}{1}$  ways.

For 2 girls, there are  $\binom{n}{2}$  ways.

$$\text{Hence } \binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}.$$

Question 13 (continued)

- (c) Consider the region bounded by the curve  $y = \frac{5}{9}x^2$ , the  $y$ -axis and the line  $y = h$ . 4

A drinking glass of height 10 cm is formed by rotating the region about the  $y$ -axis. The drinking glass is filled with water at a constant rate of  $15 \text{ cm}^3 \text{ s}^{-1}$ .

Find the rate at which the height  $h$  of water is rising with respect to time when the drinking glass is filled to half its capacity.

As  $y = \frac{5}{9}x^2$ ,  $x^2 = \frac{9y}{5}$ . For water of height  $h$ ,

$$\begin{aligned} V &= \pi \int_0^h \frac{9y}{5} dy \\ &= \frac{9\pi}{10} y^2 \Big|_0^h \\ &= \frac{9\pi}{10} (h^2 - 0) = \frac{9\pi h^2}{10}. \end{aligned}$$

At  $h = 10$ , the full capacity is  $90\pi \text{ cm}^3$ .

Now  $\frac{dV}{dh} = \frac{9\pi h}{5}$ .

Consider  $\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt}$ , by the chain rule  
 $= 15$  (given).

So  $\frac{dh}{dt} = 15 \times \frac{5}{9\pi h} = \frac{25}{3\pi h}$ .

At half capacity,  $\frac{9\pi h^2}{10} = \frac{90\pi}{2}$ , so  $h^2 = 50$  and  $h = \sqrt{50}$ , as  $h > 0$ .

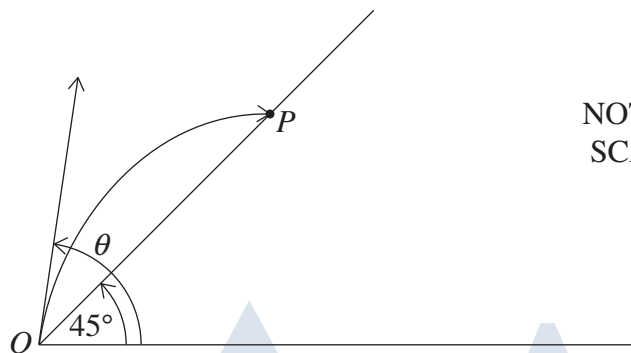
So at half capacity,  $\frac{dh}{dt} = \frac{25}{15\pi\sqrt{2}} = \frac{5}{3\pi\sqrt{2}}$ .

Hence the height rises at  $\frac{5}{3\pi\sqrt{2}} \text{ cm s}^{-1}$ .

A similar question was predicted in our **2026 HSC Masterclass** HSC-level practice paper in Question 12 (d), involving a solid of revolution container filled at a constant rate, asking for the rate at which the depth of the water changes.

Question 13 (continued)

- (d) An inclined plane passes through the origin  $O$  at an angle of  $45^\circ$  to the horizontal. A particle is projected from  $O$  with initial speed  $20 \text{ ms}^{-1}$  at an angle of  $\theta$  to the horizontal. The particle hits the plane at  $P$  as shown.



NOT TO  
SCALE

The position vector of the particle at time  $t$  seconds is

$$\mathbf{r}(t) = 20t \cos \theta \mathbf{i} + (20t \sin \theta - 5t^2) \mathbf{j}.$$

(Do NOT  
prove this.)

- (i) Show that the length,  $\ell$ , of  $OP$  is given by

3

$$\ell = 80\sqrt{2} \cos \theta (\sin \theta - \cos \theta).$$

The particle hits  $P$  when  $20t \cos \theta = \ell \cos 45^\circ$ , as  $\ell = OP$ , so  $t = \frac{\ell}{20\sqrt{2} \cos \theta}$ .

But at  $P$ ,  $\mathbf{r}(t) \cdot \mathbf{i} = \mathbf{r}(t) \cdot \mathbf{j}$ , as the plane is the line  $y = x \geq 0$ .

So

$$20t \cos \theta = 20t \sin \theta - 5t^2$$

$$t = 4(\sin \theta - \cos \theta) = \frac{\ell}{20\sqrt{2} \cos \theta}, \text{ as } t > 0.$$

Hence  $\ell = 80\sqrt{2} \cos \theta (\sin \theta - \cos \theta)$ .

Question 13 (continued)

(d) (continued)

(ii) Find  $\theta$  such that  $\ell$  is a maximum.

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By (i),

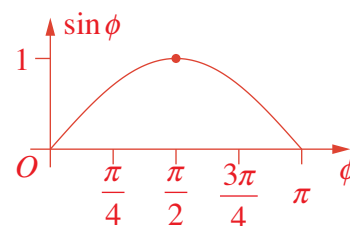
$$\begin{aligned}\ell &= 80\sqrt{2} \cos \theta (\sin \theta - \cos \theta) \\ &= 40\sqrt{2} (2 \sin \theta \cos \theta - (2 \cos^2 \theta - 1) - 1) \\ &= 40\sqrt{2} (\sin 2\theta - \cos 2\theta - 1) \\ &= 80 \left( \frac{1}{\sqrt{2}} \sin 2\theta - \frac{1}{\sqrt{2}} \cos 2\theta - \frac{1}{\sqrt{2}} \right) \\ &= 80 \left( \sin 2\theta \cos \frac{\pi}{4} - \cos 2\theta \sin \frac{\pi}{4} - \frac{1}{\sqrt{2}} \right) \\ &= 80 \left( \sin \left( 2\theta - \frac{\pi}{4} \right) - \frac{1}{\sqrt{2}} \right).\end{aligned}$$

So  $\ell$  is a maximum when  $\sin \left( 2\theta - \frac{\pi}{4} \right) = 1 = \sin \frac{\pi}{2}$ .

As  $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ ,  $\frac{\pi}{4} < 2\theta - \frac{\pi}{4} < \frac{3\pi}{4}$ ,

so  $2\theta - \frac{\pi}{4} = \frac{\pi}{2}$  and  $2\theta = \frac{3\pi}{4}$ .

Hence  $\theta = \frac{3\pi}{8}$ .



Education

Question 13 (continued)

(d) (continued)

**Alternative method**

From  $\ell = 40\sqrt{2}(\sin 2\theta - \cos 2\theta - 1)$ ,

$$\begin{aligned}\frac{d\ell}{d\theta} &= 80\sqrt{2}(\cos 2\theta + \sin 2\theta) \\ &= 0, \text{ when } \tan 2\theta = -1.\end{aligned}$$

As  $\frac{\pi}{2} < 2\theta < \pi$ ,  $2\theta = \frac{3\pi}{4}$ , so  $\theta = \frac{3\pi}{8}$ , and here  $\frac{d^2\ell}{d\theta^2} = 160\sqrt{2}(\cos 2\theta - \sin 2\theta) = -320 < 0$ , so this is indeed a maximum.

This is a very standard question, and has been covered extensively several times during our classes on *Projectile Motion* in Mathematics Extension 1.

The heuristic is recognising that the range up the incline is greatest when the angle of projection bisects the angle between the incline and the vertical. In this case,

$$\theta = \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{3\pi}{8}.$$

More general versions of this question have appeared in past HSC exams too.

Education

**Question 14** (15 marks) Answer Question 14 on pages 17–20 of the Writing Booklet.

- (a) Suppose  $f(x)$  is an increasing function and is always concave up. 2

Prove that the inverse function  $g(x)$  is concave down.

Firstly, as  $f$  is increasing, it is one-to-one, so  $g$  exists, and  $f(g(x)) \equiv x$ .

Differentiating both sides with respect to  $x$ ,  $f'(g(x))g'(x) \equiv 1$ , so  $g'(x) = \frac{1}{f'(g(x))}$ .

Once more,

$$g''(x) = -\frac{f''(g(x))g'(x)}{(f'(g(x)))^2}.$$

But  $f''(g(x)) > 0$ , as  $f$  is concave up, and  $g'(x) > 0$ , since  $f'(g(x)) > 0$ , as  $f$  is increasing.

So  $g''(x) < 0$ , as  $(f'(g(x)))^2 > 0$ .

Hence  $g(x)$  is concave down.

- (b) Find the solution to the differential equation  $\cos^2 x \frac{dy}{dx} = y$  that passes through the point  $\left(\frac{\pi}{4}, -e\right)$ . Give your answer in the form  $y = f(x)$ . 3

By separation of variables,

$$\int \frac{dy}{y} = \int \sec^2 x \, dx, \text{ as } y \neq 0$$

$$\ln|y| = \tan x + c$$

$$y = Ae^{\tan x}, \text{ where } A = \pm e^c \in \mathbb{R}.$$

For  $y\left(\frac{\pi}{4}\right) = -e$ ,  $Ae = -e$ , so  $A = -1$ .

Hence  $y = -e^{\tan x}$ , with  $f(x) = -e^{\tan x}$ .

Question 14 (continued)

- (c) Find the values of  $m$  such that the equation

3

$$3(\cos x - \sin x) + m \sin x = 5$$

has no solutions.

Consider  $3\cos x + (m - 3)\sin x = 5$ .

By auxiliary angle,  $R > 0$  and  $\alpha \in \mathbb{R}$  can be found such that

$$3\cos x + (m - 3)\sin x \equiv R\cos(x - \alpha).$$

Compare the LHS with  $RHS = (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$ , and let  $R\cos\alpha = 3$  and  $R\sin\alpha = m - 3$ .

So by squaring then adding,

$$\begin{aligned} R^2(\cos^2\alpha + \sin^2\alpha) &= R^2 = 9 + (m - 3)^2 \\ R &= \sqrt{9 + (m - 3)^2}, \text{ as } R > 0. \end{aligned}$$

The range of  $R\cos(x - \alpha)$  is  $[-R, R]$ , as  $|\cos(x - \alpha)| \leq 1$ , so if  $R \geq 5$ , then  $R\cos(x - \alpha) = 5$  will always have solutions.

Thus, for no solutions,  $R < 5$ .

So

$$\begin{aligned} 0 &< \sqrt{9 + (m - 3)^2} < 5 \\ 9 + (m - 3)^2 &< 25 \\ (m - 3)^2 &< 16 = 4^2. \end{aligned}$$

So  $|m - 3| < 4$ , that is,  $-4 < m - 3 < 4$ .

Hence  $m \in (-1, 7)$ .

Question 14 (continued)

(c) (continued)

**Alternative method**

Let  $t = \tan \frac{x}{2}$ , so  $\cos x = \frac{1-t^2}{1+t^2}$  and  $\sin x = \frac{2t}{1+t^2}$ , for  $x \neq \pi + 2k\pi$ , where  $k \in \mathbb{Z}$ .

Then

$$\begin{aligned} 3(1-t^2) + 2(m-3)t &= 5(1+t^2) \\ 4t^2 - (m-3)t + 1 &= 0. \end{aligned}$$

For no solutions,  $\Delta = (m-3)^2 - 16 < 0$ .

Hence  $-1 < m < 7$ .

It is easy, but unnecessary, to check that  $x = \pi$  is not a solution.

**Alternative method**

Let  $\underline{u} = \begin{pmatrix} 3 \\ m-3 \end{pmatrix}$  and  $\underline{v} = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$ , so the equation becomes  $\underline{u} \cdot \underline{v} = 5$ , where  $|\underline{v}| = 1$ .

As  $\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos \theta_{\underline{u}, \underline{v}}$ , for  $x \in \mathbb{R}$ ,  $|\underline{u} \cdot \underline{v}| \leq |\underline{u}| |\underline{v}| = |\underline{u}|$ . Thus, for no solutions, we set  $|\underline{u}| < 5$ . So  $9 + (m-3)^2 < 25$ .

Hence  $-1 < m < 7$ .

This is essentially the auxiliary angle method expressed using the Cauchy–Schwarz inequality, which is in the new Mathematics Extension 2 syllabus for 2027.

Education

Question 14 (continued)

- (d) Let  $P(t)$  be the population of a particular species, where  $t$  is the time in years. The population satisfies 4

$$\frac{dP}{dt} = \frac{P(5000 - P)}{5000} \text{ and } P(0) = 7000.$$

How many years does it take for the population to reach 5100? You may use the result

$$\frac{5000}{P(5000 - P)} = \frac{1}{P} + \frac{1}{5000 - P}. \quad (\text{Do NOT prove this.})$$

Note that  $P(t) = 5000$  is a constant solution. But  $P(0) = 7000 > 5000$ , so  $P(t) > 5000$  for  $t \geq 0$ .

Separating variables,  $\int \frac{5000 dP}{P(5000 - P)} = \int dt.$

By the given result,

$$\int \left( \frac{1}{P} + \frac{1}{5000 - P} \right) dP = \int dt$$

$$\ln \left( \frac{P}{P - 5000} \right) = t + c, \text{ as } P > 5000.$$

When  $t = 0$ ,  $P = 7000$ , so  $c = \ln \left( \frac{7000}{2000} \right) = \ln \left( \frac{7}{2} \right).$

So for  $P = 5100$ ,

$$t = \ln \left( \frac{5100}{100} \right) - \ln \left( \frac{7}{2} \right)$$

$$= \ln \left( \frac{102}{7} \right).$$

Hence it takes  $\ln \left( \frac{102}{7} \right)$  years.

Justification for why  $P > 5000$  should be given. Alternatively, the absolute values can be absorbed by another constant, say  $A$ , using the method in **Question 14 (b)**.

Question 14 (continued)

- (e) Show that  $y = x \tan^{-1} x$  has only one stationary point.

3

Consider  $y' = \tan^{-1} x + \frac{x}{1+x^2}$ .

We show that  $y = x \tan^{-1} x$  has only one stationary point at  $x = 0$ .

At  $x = 0$ ,  $y' = \tan^{-1} 0 + \frac{0}{1+0^2} = 0 + 0 = 0$ , so at  $x = 0$  there is a stationary point.

For  $x > 0$ ,  $y' = \tan^{-1} x + \frac{x}{1+x^2} > 0 + 0$ , as  $\tan^{-1} x > 0$  for  $x > 0$ , and  $\frac{x}{1+x^2} > 0$ , since  $1+x^2 > 0$ .

So for  $x > 0$ ,  $y' > 0$ , and thus  $y' \neq 0$  for any  $x > 0$ .

$$\begin{aligned} \text{Now } y(-x) &= -x \tan^{-1}(-x) \\ &= (-x)(-\tan^{-1} x), \text{ as } \tan^{-1}(\cdot) \text{ is odd} \\ &= x \tan^{-1} x = y(x), \end{aligned}$$

so  $y = x \tan^{-1} x$  is even.

As  $y = x \tan^{-1} x$  is even,  $y' < 0$  for  $x < 0$ , so  $y' \neq 0$  for any  $x < 0$ .

Thus  $y' = 0$  at  $x = 0$  ONLY.

Hence  $y = x \tan^{-1} x$  has only one stationary point, namely at  $x = 0$ .

**Alternative method**

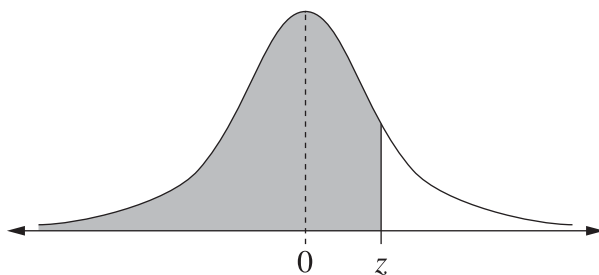
As  $y'' = \frac{1}{1+x^2} + \frac{(1+x^2) - 2x^2}{(1+x^2)^2} = \frac{2}{(1+x^2)^2} > 0$  for all  $x \in \mathbb{R}$ ,  $y'$  is strictly increasing.

Hence, as  $y'(0) = 0$ ,  $y' = 0$  has exactly one solution,  $x = 0$ .

The alternative method uses the same idea as Question 13 (a) of our **2025 HSC Premiere**, where the second derivative was used to determine the sign of the first derivative. Though our question was more difficult, because it also layered in the concept of inverse functions as well as conditions for invertibility.

Use the information below to answer Question 13 (a).

Table of values  $P(Z \leq z)$  for the normal distribution  $N(0, 1)$



| $z$ | 0.00   | 0.01   | 0.02   | 0.03   | 0.04   | 0.05   | 0.06   | 0.07   | 0.08   | 0.09   |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |



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